Available online at www.sciencedirect.com

SCIENCE DIRECT®

Stochastic Processes and their Applications 112 (2004) 53–78

stochastic processes and their applications

www.elsevier.com/locate/spa

Ruin probabilities and penalty functions with stochastic rates of interest

Jun Cai*

Department of Statistics and Actuarial Science, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, Canada N2L 3G1

Received 11 April 2003; received in revised form 11 December 2003; accepted 22 January 2004

Abstract

Assume that a compound Poisson surplus process is invested in a stochastic interest process which is assumed to be a Lévy process. We derive recursive and integral equations for ruin probabilities with such an investment. Lower and upper bounds for the ultimate ruin probability are obtained from these equations. When the interest process is a Brownian motion with drift, we give a unified treatment to ruin quantities by studying the expected discounted penalty function associated with the time of ruin. An integral equation for the penalty function is given. Smooth properties of the penalty function are discussed based on the integral equation. Errors in a known result about the smooth properties of the ruin probabilities are corrected. Using a differential argument and moments of exponential functionals of Brownian motions, we derive an integro-differential equation satisfied by the penalty function. Applications of the integro-differential equation are given to the Laplace transform of the time of ruin, the deficit at ruin, the amount of claim causing ruin, etc. Some known results about ruin quantities are recovered from the generalized penalty function.

Keywords: Ruin theory; Penalty function; Stochastic rates of interest; Integral equation; Integro-differential equation; Compound Poisson process; Lévy process; Subordinator; Brownian motion

1. Introduction

Let $\{U_t = ct - \sum_{k=1}^{N(t)} Y_k,\ t \geqslant 0\}$ be a compound Poisson surplus process, where $c > 0$ is the rate of premium, $\{Y_n,\ n = 1, 2, \ldots\}$ is a sequence of independent and identically distributed (i.i.d.) nonnegative random variables, denoting claim sizes and $N(t)$ is a Poisson process with rate $\lambda > 0$, representing the number of claims up to time t.

* Tel.: +1-519-8884567; fax: +1-519-7461875.
E-mail address: jcai@icarus.math.uwaterloo.ca (J. Cai).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2004.01.007

Suppose that an insurer would invest its surplus in a stochastic interest process $\{R_t,\ t \geqslant 0\}$, which is assumed to be a Lévy process, independent of the process $\{U_t,\ t \geqslant 0\}$. That is to say that one unit invested at time 0 will become e^{R_t} at time t, or equivalently, one unit at time t has a present value e^{-R_t} at time 0.

Let X_t denote the surplus at time t of the insurer with such as investment and an initial surplus u, thus,

$$X_t = e^{R_t}\left(u + \int_0^t e^{-R_s}\, dU_s\right), \quad t \geqslant 0,\ X_0 = u, \tag{1.1}$$

which is a simplified version of the surplus process considered by Kalashnikov and Norberg (2002) or Paulsen (2002).

Denote the time of ruin of the surplus process $\{X_t,\ t \geqslant 0\}$ by T, i.e. $T = \inf\{t: X_t < 0\}$ and ∞ if $X_t \geqslant 0$ for all $t \geqslant 0$, and the ultimate ruin probability with an initial surplus u by $\psi(u)$, then $\psi(u) = \Pr\{T < \infty\} = \Pr\{\inf_{t>0} X_t < 0\}$.

Let T_k be the time of the kth claim, $k = 1, 2, \ldots$ and $\{S_n,\ n = 0, 1, 2, \ldots\}$ be the embedded discrete time process of $\{X_t,\ t \geqslant 0\}$ at claim times, namely, $S_n = X_{T_n},\ n = 0, 1, 2, \ldots,\ S_0 = u$. Further, $\{S_n,\ n = 0, 1, 2, \ldots\}$ satisfies the following recursive equation:

$$S_n = A_n S_{n-1} + cB_n - Y_n, \quad n = 1, 2, \ldots,\ S_0 = u, \tag{1.2}$$

where $\{(A_n, cB_n - Y_n),\ n = 1, 2, \ldots\}$ is a sequence of i.i.d. two-dimensional random variables with

$$A_1 = e^{R_{T_1}} \quad \text{and} \quad B_1 = \int_0^{T_1} e^{R_s}\, ds. \tag{1.3}$$

For the generalization of (1.2) and (1.3), see, for example, Kalashnikov and Norberg (2002).

It is not hard to see that (1.2) implies

$$S_n = u \prod_{k=1}^{n} A_k + \sum_{k=1}^{n} \left((cB_k - Y_k) \prod_{t=k+1}^{n} A_t \right), \quad n = 1, 2, \ldots, \tag{1.4}$$

where $\prod_{t=a}^{b} = 1$ and $\sum_{t=a}^{b} = 0$ if $a > b$.

With the surplus process X_t, ruin can occur only at claim times, so

$$\psi(u) = \Pr\left\{ \bigcup_{k=1}^{\infty} (S_k < 0) \right\}.$$

Further, let

$$\begin{aligned}\psi_n(u) &= \Pr\left\{ \bigcup_{k=1}^{n} (S_k < 0) \right\} \\ &= \Pr\left\{ \bigcup_{k=1}^{n} \left(u \prod_{t=1}^{k} A_t + \sum_{j=1}^{k} (cB_j - Y_j) \prod_{t=j+1}^{k} A_t < 0 \right) \right\}\end{aligned}$$

be the probability that ruin occurs before or on the nth claim with an initial surplus u. Clearly,

$$0 \leqslant \psi_1(u) \leqslant \psi_2(u) \leqslant \cdots \leqslant \psi_n(u) \leqslant \cdots \quad \text{and} \quad \psi(u) = \lim_{n\to\infty} \psi_n(u).$$

Except for the ruin probabilities, other important ruin quantities in ruin theory include the Laplace transform of the time of ruin, $E(\mathrm{e}^{-\alpha T})$; the surplus immediately before ruin, denoted by X_{T-}; the deficit at ruin, $|X_T|$; and the amount of claim causing ruin, $X_{T-} + |X_T|$; etc. A unified method to study these ruin quantities is to consider the (expected discounted) penalty function associated with the time of ruin by defining

$$\Phi_\alpha(u) = E[g(X_{T-}, |X_T|)\mathrm{e}^{-\alpha T} I(T < \infty)], \tag{1.5}$$

where $g(x, y),\ x \geqslant 0,\ y \geqslant 0$, is a nonnegative function such that $\Phi_\alpha(u)$ exists; $\alpha \geqslant 0$; and $I(C)$ is the indicator function of a set C.

A simple and sufficient condition on g for $\Phi_\alpha(u) < \infty$ is that g is a bounded function. With suitable choices of g, $\Phi_\alpha(u)$ will yield different ruin quantities. For example, if $g = 1$ and $\alpha = 0$, then $\Phi_\alpha(u) = \psi(u)$ is the ruin probability; if $g = 1$ and $\alpha > 0$, then $\Phi_\alpha(u) = E(\mathrm{e}^{-\alpha T} I(T < \infty)) = E(\mathrm{e}^{-\alpha T})$ gives the Laplace transform of the time of ruin since $\mathrm{e}^{-\alpha T} = \mathrm{e}^{-\alpha T} I(T < \infty) + \mathrm{e}^{-\alpha T} I(T = \infty) = \mathrm{e}^{-\alpha T} I(T < \infty)$ if $\alpha > 0$; if $g(x_1, x_2) = I(x_2 \leqslant y)$ and $\alpha = 0$, then $\Phi_\alpha(u) = \Pr\{|X_T| \leqslant y,\ T < \infty\}$ denotes the distribution function of the deficit at ruin; if $g(x_1, x_2) = I(x_1 + x_2 \leqslant y)$ and $\alpha = 0$, then $\Phi_\alpha(u) = \Pr\{X_{T-} + |X_T| \leqslant y,\ T < \infty\}$ represents the distribution function of the amount of claim causing ruin. See Gerber and Shiu (1997) for $\Phi_\alpha(u)$ in the compound Poisson risk model and Cai and Dickson (2002) for it in the compound Poisson risk model with a constant interest rate.

One of the common research methods used in ruin theory is first to derive integral and integro-differential equations for ruin quantities, and then try to solve these equations and obtain explicit solutions. When a surplus process is invested in a stochastic interest process, explicit solutions are rarely available. However, with the integral and integro-differential equations, we can derive some analytic properties of the ruin quantities. In addition, we can use numerical methods to solve the equations and obtain numerical solutions.

Norberg (1999) derived differential equations for ruin probabilities in various stochastic investment models under the assumption that the ruin probabilities are twice continuously differentiable. Paulsen and Gjessing (1997) considered a special Lévy process for R_t and showed that if the Laplace transform of the time of ruin (or the ruin probability) is twice continuously differentiable with a bounded first derivative, then the Laplace transform of the time of ruin (or the ruin probability) is the solution of a second-order integro-differential equation. Further, Paulsen (2002) derived an asymptotic formula for the ruin probability $\psi(u)$ using the second-order integro-differential equation.

However, it is not easy to check if the ruin probabilities or other ruin quantities are twice or more times differentiable and if they have first- or high-order derivatives. Recently, Wang and Wu (2001) have proved that the non-ruin probability, hence the ruin probability, is indeed twice differentiable under some conditions when R_t is a Brownian motion with drift. However, there are some mistakes in the conditions and their proof, which will be pointed out and corrected in Remark 3.1 of this paper. In addition, Wang and Wu (2001) have showed that the distribution functions of the deficit at ruin and the supremum of the surplus before ruin are twice differentiable under some conditions and satisfy some integro-differential equations.

In this paper, we give a unified treatment to the ruin quantities when R_t is a Brownian motion with drift by studying the generalized penalty function $\Phi_\alpha(u)$. We derive integral and integro-differential equations for $\Phi_\alpha(u)$ and show that $\Phi_\alpha(u)$ is twice continuously differentiable with bounded first and second derivatives under some conditions. When R_t is a Lévy process, we concentrate on ruin probabilities.

This paper is organized as follows. In Section 2, we derive a recursive integral equation for $\psi_n(u)$ and an integral equation for $\psi(u)$ when R_t is a Lévy process and consider applications of these equations. In particular, we give a lower bound for $\psi(u)$ when R_t is a Lévy process and an upper bound for $\psi(u)$ when R_t is a nonnegative Lévy process. In Sections 3 and 4, we assume that R_t is a Brownian motion with drift. We derive an integral equation satisfied by $\Phi_\alpha(u)$. Using the integral equation, we discuss differentiability of $\Phi_\alpha(u)$. In Section 4, using a differential argument and moments of exponential functionals of Brownian motions and differentiability of $\Phi_\alpha(u)$, we derive an integro-differential equation for $\Phi_\alpha(u)$. In Section 5, examples are given to illustrate applications of the integro-differential equation, including the Laplace transform of the time of ruin, the deficit at ruin, the amount of claim causing ruin, etc. Some known results about ruin quantities are recovered from the generalized penalty function.

2. Integral equations and bounds

Let (A_1, B_1) given in (1.3) have a joint density function $p(x,w)$, $x>0$, $w>0$ and $F(x)=1-\bar{F}(x)$ be the distribution function of Y_1.

The following result gives a recursive integral equation for $\psi_n(u)$.

Theorem 2.1. *For $n=1,2,\ldots$ and $u \geqslant 0$,*

$$\psi_{n+1}(u) = \int_0^\infty \int_0^\infty \bar{F}(ux+cw)\,p(x,w)\,\mathrm{d}x\,\mathrm{d}w + \int_0^\infty \int_0^\infty \int_0^{ux+cw} \psi_n(ux+cw-y)\,\mathrm{d}F(y)\,p(x,w)\,\mathrm{d}x\,\mathrm{d}w \tag{2.1}$$

with

$$\psi_1(u) = \int_0^\infty \int_0^\infty \bar{F}(ux+cw)\,p(x,w)\,\mathrm{d}x\,\mathrm{d}w = E[\bar{F}(uA_1+cB_1)].$$

Proof. From (1.2) and independence of Y_1 and (A_1, B_1), we have

$$\psi_1(u) = \Pr\{S_1<0\} = \Pr\{Y_1 > uA_1 + cB_1\} = \int_0^\infty \int_0^\infty \bar{F}(ux+cw)\,p(x,w)\,\mathrm{d}x\,\mathrm{d}w.$$

Given $Y_1=y$, $A_1=x$, and $B_1=w$, if $y>ux+cw$, then

$$\Pr\{S_1<0 \mid Y_1=y,\ A_1=x,\ B_1=w\}=1,$$

which implies $\Pr\{\bigcup_{k=1}^{n+1}(S_k<0) \mid Y_1=y,\ A_1=x,\ B_1=w\}=1$.

If $0 \leqslant y \leqslant ux+cw$, then $\Pr\{S_1 < 0 \mid Y_1 = y,\ A_1 = x,\ B_1 = w\} = 0$, which implies by (1.4) that for $0 \leqslant y \leqslant ux+cw$,

$$
\begin{aligned}
&\Pr\left\{\bigcup_{k=1}^{n+1} (S_k < 0) \mid Y_1 = y,\ A_1 = x,\ B_1 = w\right\} \\
&= \Pr\left\{\bigcup_{k=2}^{n+1} (S_k < 0) \mid Y_1 = y,\ A_1 = x,\ B_1 = w\right\} \\
&= \Pr\left\{\bigcup_{k=2}^{n+1} \left((ux+cw-y)\prod_{t=2}^{k} A_t + \sum_{j=2}^{k} (cB_j - Y_j) \prod_{t=j+1}^{k} A_t < 0\right)\right\} \\
&= \psi_n(ux+cw-y).
\end{aligned}
$$

Therefore, by conditioning on Y_1, A_1, and B_1, we obtain (2.1). □

An integral equation for $\psi(u)$ follows immediately from (2.1).

Corollary 2.1. *For* $u \geqslant 0$,

$$
\begin{aligned}
\psi(u) = &\int_0^\infty \int_0^\infty \bar{F}(ux+cw)\, p(x,w)\, \mathrm{d}x\, \mathrm{d}w \\
&+ \int_0^\infty \int_0^\infty \int_0^{ux+cw} \psi(ux+cw-y)\, \mathrm{d}F(y)\, p(x,w)\, \mathrm{d}x\, \mathrm{d}w. \qquad (2.2)
\end{aligned}
$$

Proof. Eq. (2.2) follows from $\lim_{n\to\infty} \psi_n(u) = \psi(u)$, the monotone convergence theorem, and (2.1). □

Using the recursive and integral equations (2.1) and (2.2), we can derive lower and upper bounds for $\psi(u)$. First, using the integral equation (2.2), we give a general lower bound for $\psi(u)$ when R_t is a Lévy process.

Corollary 2.2. *For any* $u \geqslant 0$,

$$
\psi(u) \geqslant \frac{\int_0^\infty \int_0^\infty \bar{F}(ux+cw)\, p(x,w)\, \mathrm{d}w\, \mathrm{d}x}{1 - \int_0^1 \int_0^{(u-ux)/c} F(ux+cw)\, p(x,w)\, \mathrm{d}w\, \mathrm{d}x}. \qquad (2.3)
$$

Proof. Since $\psi(u)$ is a decreasing function, we know that if $0 \leqslant ux+cw \leqslant u$, then

$$
\int_0^{ux+cw} \psi(ux+cw-y)\, \mathrm{d}F(y) \geqslant \psi(ux+cw)F(ux+cw) \geqslant \psi(u)F(ux+cw).
$$

Thus,

$$\int_0^\infty \int_0^\infty \int_0^{ux+cw} \psi(ux+cw-y)\,\mathrm{d}F(y)\,p(x,w)\,\mathrm{d}x\,\mathrm{d}w$$

$$\geqslant \iint_{0\leqslant ux+cw\leqslant u,\ x>0,\ w>0} \int_0^{ux+cw} \psi(ux+cw-y)\,\mathrm{d}F(y)\,p(x,w)\,\mathrm{d}x\,\mathrm{d}w$$

$$\geqslant \psi(u) \iint_{0\leqslant ux+cw\leqslant u,\ x>0,\ w>0} F(ux+cw)\,p(x,w)\,\mathrm{d}x\,\mathrm{d}w$$

$$= \psi(u) \int_0^1 \int_0^{(u-ux)/c} F(ux+cw)\,p(x,w)\,\mathrm{d}w\,\mathrm{d}x,$$

which, together with (2.2), implies that (2.3) holds. □

Next, using the recursive equation (2.1), we derive an upper bound for $\psi(u)$ when R_t is a nonnegative Lévy process such as subordinators, gamma processes, nonnegative linear processes, etc.

Corollary 2.3. *Suppose that* $R_t \geqslant 0$ *and* $R>0$ *is a constant satisfying*

$$E(\exp\{R(Y_1 - cB_1)\}) = 1. \tag{2.4}$$

Then, for any $u \geqslant 0$,

$$\psi(u) \leqslant \alpha E(\exp\{RY_1\}) E(\exp\{-R(uA_1 + cB_1)\}) \tag{2.5}$$

$$\leqslant \alpha \mathrm{e}^{-Ru}, \tag{2.6}$$

where $(\alpha)^{-1} = \inf_{t\geqslant 0}\{\int_t^\infty \exp\{Ry\}\,\mathrm{d}F(y)/(\exp\{Rt\}\bar{F}(t))\}$ *with* $0 \leqslant \alpha \leqslant 1$.

Proof. For any $x \geqslant 0$, we have

$$\bar{F}(x) = \left(\frac{\int_x^\infty \exp\{Ry\}\,\mathrm{d}F(y)}{\exp\{Rx\}\bar{F}(x)}\right)^{-1} \exp\{-Rx\} \int_x^\infty \exp\{Ry\}\,\mathrm{d}F(y)$$

$$\leqslant \alpha \exp\{-Rx\} \int_x^\infty \exp\{Ry\}\,\mathrm{d}F(y) \tag{2.7}$$

$$\leqslant \alpha \exp\{-Rx\} E(\exp\{RY_1\}). \tag{2.8}$$

Then, for any $u \geqslant 0$, by (2.8),

$$\psi_1(u) = E[\bar{F}(uA_1 + cB_1)] \leqslant \alpha E(\exp\{RY_1\}) E(\exp\{-R(uA_1 + cB_1)\}).$$

Under an inductive hypothesis, we assume for any $u \geqslant 0$,

$$\psi_n(u) \leqslant \alpha E(\exp\{RY_1\}) E(\exp\{-R(uA_1 + cB_1)\}). \tag{2.9}$$

It is obvious that $A_1 = \exp\{R_{T_1}\} \geqslant 1$ if $R_t \geqslant 0$. Thus, for $0 \leqslant y \leqslant ux + cw$, by (2.9), $B_1 \geqslant 0$, (2.4), and independence of Y_1 and B_1, we have

$$\begin{aligned}
\psi_n(ux + cw - y) &\leqslant \alpha E(\exp\{RY_1\})E(\exp\{-R[(ux + cw - y)A_1 + cB_1]\}) \\
&\leqslant \alpha E(\exp\{RY_1\})E(\exp\{-R(ux + cw - y) - RcB_1\}) \\
&= \alpha E(\exp\{R(Y_1 - cB_1)\})\exp\{-R(ux + cw - y)\} \\
&= \alpha \exp\{-R(ux + cw - y)\}. \qquad (2.10)
\end{aligned}$$

Thus, by (2.1), (2.7), and (2.10), we get

$$\begin{aligned}
\psi_{n+1}(u) &\leqslant \alpha \int_0^\infty \int_0^\infty \exp\{-R(ux + cw)\} \int_{ux+cw}^\infty \exp\{Ry\}\, \mathrm{d}F(y)\, p(x, w)\, \mathrm{d}x\, \mathrm{d}w \\
&\quad + \alpha \int_0^\infty \int_0^\infty \exp\{-R(ux + cw)\} \int_0^{ux+cw} \exp\{Ry\}\, \mathrm{d}F(y)\, p(x, w)\, \mathrm{d}x\, \mathrm{d}w \\
&= \alpha \int_0^\infty \int_0^\infty \exp\{-R(ux + cw)\} \int_0^\infty \exp\{Ry\}\, \mathrm{d}F(y)\, p(x, w)\, \mathrm{d}x\, \mathrm{d}w \\
&= \alpha E\exp\{RY_1\}E\exp\{-R(uA_1 + cB_1)\}.
\end{aligned}$$

Hence, for all $n = 1, 2, \ldots$, (2.9) holds. Therefore, (2.5) follows from letting $n \to \infty$ in (2.9) and $\lim_{n\to\infty} \psi_n(u) = \psi(u)$.

Eq. (2.6) follows from (2.5), $A_1 \geqslant 1$, and (2.4). $\square$

An improved upper bound in Corollary 2.3 can be obtained when F is new worse than used in convex ordering (NWUC), where F is said to be NWUC if

$$\int_{x+y}^\infty \bar{F}(t)\, \mathrm{d}t \geqslant \bar{F}(x) \int_y^\infty \bar{F}(t)\, \mathrm{d}t$$

for all $x \geqslant 0$ and $y \geqslant 0$. For the details of NWUC, see Willmot and Lin (2001).

Corollary 2.4. *Under the conditions of Corollary* 2.3, *if F is NWUC, then*

$$\psi(u) \leqslant E(\exp\{-R(uA_1 + cB_1)\}), \quad u \geqslant 0. \qquad (2.11)$$

Proof. By Proposition 6.1.1 of Willmot and Lin (2001), we know that if F is NWUC then $\alpha = [E(\exp\{RY\})]^{-1}$. Thus (2.11) follows from (2.5). $\square$

An example of nonnegative Lévy processes is $R_t = \delta t$, $\delta > 0$. With such an interest process, $A_1 = \exp\{T_1\}$ and $B_1 = (\exp\{T_1\} - 1)/\delta$. This model has been considered by Sundt and Teugels (1995), Cai and Dickson (2003), and references therein. In particular, Theorem 4.2 of Cai and Dickson (2003) is a special case of Corollary 2.3 above. In addition, the technique used in this section has appeared in Cai (2002), in which ruin probabilities in discrete time risk models with stochastic interest rates are considered.

When $R_t \geqslant 0$, it is obvious that the ruin probability $\psi(u)$ is less than $\psi_0(u)$, the ruin probability in the compound Poisson risk model without interest or $R_t = 0$. For $\psi_0(u)$, the well-known Lundberg upper bound states that if $cET_1 = c/\lambda > EY_1$ and there is a constant $R_0 > 0$ satisfying

$$E(\exp\{R_0(Y_1 - cT_1)\}) = 1, \tag{2.12}$$

then $\psi_0(u) \leqslant \exp\{-R_0 u\}$, $u \geqslant 0$, see, for example, Asmussen (2000) or Rolski et al. (1999). Therefore, any interesting upper bound for $\psi(u)$, saying $\psi(u) \leqslant \Lambda(u)$, $u \geqslant 0$, should be less than the Lundberg upper bound for $\psi_0(u)$, namely, $\Lambda(u) \leqslant \exp\{-R_0 u\}$, $u \geqslant 0$. We can show that the upper bounds in Corollaries 2.3 and 2.4 are less than the Lundberg upper bound. To see that, it is sufficient to prove $R \geqslant R_0$. Indeed, we have the following result.

Proposition 2.1. *If $cE(T_1) = c/\lambda > EY_1$ and $R > 0$ in (2.4) and $R_0 > 0$ in (2.12) exist, then $R \geqslant R_0$. In particular, if B_1 is not degenerate at 0, then $R > R_0$.*

Proof. Let $f(r) = E(\exp\{r(Y_1 - cB_1)\}) - 1$ and $g(r) = E(\exp\{r(Y_1 - cT_1)\}) - 1$. We have $f''(r) \geqslant 0$, which implies that $f(r)$ is a convex function. In addition, $f(0) = 0$ and $f'(0) = EY_1 - cEB_1 \leqslant EY_1 - cET_1 < 0$ since $B_1 \geqslant T_1$ when $R_t \geqslant 0$. Similarly, $g(r)$ is also a convex function with $g(0) = 0$ and $g'(0) = E(Y_1 - cT_1) < 0$. Therefore, if $R > 0$ and $R_0 > 0$ exist, then they are unique positive roots of $f(r)$ and $g(r)$, respectively, on $(0, \infty)$. Further, if $r > 0$ such that $g(r) \geqslant 0$, then $r \geqslant R_0$. However, we know $\exp\{R(Y_1 - cB_1)\} \leqslant \exp\{R(Y_1 - cT_1)\}$. Thus, $1 = E(\exp\{R(Y_1 - cB_1)\}) \leqslant E(\exp\{R(Y_1 - cT_1)\})$, or $g(R) \geqslant 0$, which implies that $R \geqslant R_0$. In particular, if B_1 is not degenerate at 0, then $1 = E(\exp\{R(Y_1 - cB_1)\}) < E(\exp\{R(Y_1 - cT_1)\})$, or $g(R) = E(\exp\{R(Y_1 - cB_1)\}) - 1 > 0$, which implies that $R > R_0$. □

In general, it is very difficult to obtain any explicit solutions for $\psi(u)$ when R_t is a stochastic Process. However, using (2.1), we can calculate $\psi_n(u)$ recursively, hence approximate $\psi(u)$ numerically. In addition, we can use the bounds for $\psi(u)$ to estimate $\psi(u)$. But, in order to apply these results, we need to know the law of (A_1, B_1). Thanks to the works of Yor (1992) and Carmona et al. (1994), we know the law of (A_1, B_1) when R_t belongs to a class of certain Lévy processes. In fact, Carmona et al. (1994) have derived the law of (A_1, B_1) when R_t is a Levy process with the following Lévy exponent:

$$E(\exp\{mR_t\}) = \exp\left\{tm\,\frac{ma+b}{cm+d}\right\}, \quad t \geqslant 0.$$

The examples of such Levy processes include $R_t = \delta t$, $R_t = \delta t + \sigma B_t$, and the compound Poisson processes with exponential jumps and drift, where B_t is a standard Brownian motion.

We give the law of (A_1, B_1) when $R_t = \delta t + \sigma B_t$, which will be studied in Sections 3 and 4.

Define

$$B_t^{(v)} = vt + B_t \quad \text{and} \quad A_t^{(v)} = \int_0^t \exp\{2(vs + B_s)\}\,\mathrm{d}s.$$

It is obvious that if (X, Y) has a joint density function $f(x, y)$, $x > 0$, $y > 0$, then $(X^2, (4/\sigma^2)Y)$ has a joint density function

$$\frac{\sigma^2}{8\sqrt{x}} f\left(\sqrt{x}, \frac{\sigma^2}{4} y\right), \quad x > 0, \ y > 0. \tag{2.13}$$

Thus, we have the following result.

Proposition 2.2. *Let* $R_t = \delta t + \sigma B_t$. *Then* $(\exp\{R_{T_1}\}, \int_0^{T_1} \exp\{R_s\}\, \mathrm{d}s)$ *has the density function*

$$p(x, y) = \frac{\lambda}{2(\sqrt{x})^{3+\gamma-2\delta/\sigma^2}}\, p^{\gamma}_{\sigma^2 y/4}(1, \sqrt{x}), \quad x > 0, \ y > 0, \tag{2.14}$$

where $\gamma = (2/\sigma)\sqrt{2\lambda + \delta^2/\sigma^2}$ *and*

$$p^{\beta}_{y}(1, x) = x^{\beta} \left(\frac{x}{y}\right) \exp\left(-\frac{1 + x^2}{2y}\right) I_{\beta}\left(\frac{x}{y}\right) \tag{2.15}$$

and $I_\beta(z)$ *is the modified Bessel function of index* β.

Proof. Let S_θ be an exponential random variable with mean $2/\theta^2$ and is independent of the Brownian motion B_t. Then, Theorem 2 of Yor (1992) states that $(\exp\{B^{(v)}_{S_\theta}\}, A^{(v)}_{S_\theta})$ has the following density function:

$$\tilde{p}(x, y) = \frac{\theta^2}{2x^{2+\gamma-v}}\, p^{\gamma}_{y}(1, x), \quad x > 0, \ y > 0, \tag{2.16}$$

where $\gamma = (\theta^2 + v^2)^{1/2}$ and $p^{\gamma}_{y}(1, x)$ is given by (2.15).

By the scaling property of Brownian motions, we know that $\tilde{B}_t = (\sigma/2) B_{4t/\sigma^2}$ is also a Brownian motion. Thus, by setting $s = 4t/\sigma^2$ and noting $\tilde{B}_{\sigma^2 t/4} = (\sigma/2) B_t$, we have

$$\begin{aligned}
&\left(\exp\{R_{T_1}\}, \int_0^{T_1} \exp\{R_s\}\, \mathrm{d}s\right) \\
&\quad = \left(\exp\{\delta T_1 + \sigma B_{T_1}\}, \int_0^{T_1} \exp\left\{2\left[\left(\frac{\delta}{2}\right) s + \left(\frac{\sigma}{2}\right) B_s\right]\right\} \mathrm{d}s\right) \\
&\quad = \left(\exp\left\{2\left(\frac{\delta}{2} T_1 + \frac{\sigma}{2} B_{T_1}\right)\right\}, \frac{4}{\sigma^2} \int_0^{\sigma^2 T_1/4} \exp\left\{2\left[\left(\frac{\delta}{2}\right)\left(\frac{4t}{\sigma^2}\right)\right.\right.\right. \\
&\qquad \left.\left.\left. + \left(\frac{\sigma}{2}\right) B_{4t/\sigma^2}\right]\right\} \mathrm{d}t\right) \\
&\quad = \left(\exp\left\{2\left[\left(\frac{2\sigma}{\sigma^2}\right)\left(\frac{\sigma^2 T_1}{4}\right) + \tilde{B}_{\sigma^2 T_1/4}\right]\right\},\right.
\end{aligned}$$

$$\frac{4}{\sigma^2}\int_0^{\sigma^2 T_1/4} \exp\left\{2\left[\left(\frac{2\delta}{\sigma^2}\right)t+\tilde{B}_t\right]\right\} \mathrm{d}t\right)$$

$$=^{\mathrm{d}} \left(\left[\exp\left\{B^{(2\delta/\sigma^2)}_{\sigma^2 T_1/4}\right\}\right]^2, \frac{4}{\sigma^2} A^{(2\delta/\sigma^2)}_{\sigma^2 T_1/4}\right), \tag{2.17}$$

where $=^{\mathrm{d}}$ means equality in distribution.

Further, $\sigma^2 T_1/4$ is an exponential random variable with mean $\sigma^2/(4\lambda)$. Thus, let $\theta^2/2 = 4\lambda/\sigma^2$ or $\theta^2 = 8\lambda/\sigma^2$ and $v = 2\delta/\sigma^2$, by (2.17), (2.16), and (2.13), we obtain

$$p(x,y) = \frac{\sigma^2}{8\sqrt{x}}\tilde{p}\left(\sqrt{x}, \frac{\sigma^2}{4}y\right), \quad x>0, \ y>0,$$

which implies that (2.14) holds. □

3. Integral equations and differentiability

In Sections 3 and 4, we consider the case when $R_t = \delta t + \sigma B_t$ and assume that $\delta > 0$ and $cET_1 = c/\lambda > EY_1$. Thus, we have

$$\psi(u) < 1, \ u \geqslant 0 \quad \text{and} \quad \psi(+\infty) = \lim_{u\to\infty}\psi(u) = 0, \tag{3.1}$$

see, for example, Paulsen (1993).

In this section, we derive an integral equation for $\Phi_\alpha(u)$ and discuss differentiability of $\Phi_\alpha(u)$ based on the integral equation.

It is clear that if (X, Y) has a joint density function $f(x,y)$, $-\infty < x < \infty$, $y > 0$. Then $(\exp\{2X\}, 4Y/\sigma^2)$ has the following joint density function:

$$\frac{\sigma^2}{8x} f\left(\frac{\ln x}{2}, \frac{\sigma^2 y}{4}\right), \quad x>0, \ y>0. \tag{3.2}$$

Thus, we obtain the following result.

Proposition 3.1. *Let $R_t = \delta t + \sigma B_t$ and $g_t(x,w)$ denote the joint density function of*

$$\left(\exp\{R_t\}, \int_0^t \exp\{R_s\}\,\mathrm{d}s\right).$$

Then,

$$g_t(x,w) = \frac{x^{(\delta/\sigma^2-1)}}{2w}\exp\left(-\frac{\delta^2 t}{2\sigma^2} - \frac{2(1+x)}{\sigma^2 w}\right)\theta_{4\sqrt{x}/(\sigma^2 w)}\left(\frac{\sigma^2 t}{4}\right),$$

$$x>0, \ w>0, \tag{3.3}$$

where $\theta_r(u) = \{r/(2\pi^3 u)^{1/2}\}\exp\{\pi^2/(2u)\}\chi_r(u)$ with

$$\chi_r(u) = \int_0^\infty \exp\left\{-\frac{y^2}{2u} - r(\cosh y)\right\}(\sinh y)\sin\left(\frac{\pi y}{u}\right)\,\mathrm{d}y.$$

Proof. Similar to (2.17), we have

$$\left(\exp\{R_t\}, \int_0^t \exp\{R_s\}\,\mathrm{d}s\right) =^{\mathrm{d}} \left(\left[\exp\left\{B^{(2\delta/\sigma^2)}_{\sigma^2 t/4}\right\}\right]^2, \frac{4}{\sigma^2} A^{(2\delta/\sigma^2)}_{\sigma^2 t/4}\right). \tag{3.4}$$

Denote $\Pr\{A_t^{(\delta)} \in \mathrm{d}w \mid B_t^{(\delta)} = x\} = a_t(x,w)\,\mathrm{d}w$. By Proposition 2 of Yor (1992), we have

$$\frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\} a_t(x,w) = \frac{1}{w}\exp\left(-\frac{1+\mathrm{e}^{2x}}{2w}\right)\theta_{\mathrm{e}^x/w}(t),$$
$$-\infty < x < \infty, \quad w > 0,$$

which is independent of the parameter δ.

Since $B_t^{(\delta)}$ has the density function $(1/\sqrt{2\pi t})\exp\{-(x-\delta t)^2/(2t)\}$, $-\infty < x < \infty$, we know that $(B_t^{(\delta)}, A_t^{(\delta)})$ has the joint density function

$$\frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(x-\delta t)^2}{2t}\right\} a_t(x,w) = \exp\left\{\delta x - \frac{\delta^2 t}{2}\right\} \frac{1}{w}\exp\left(-\frac{1+\mathrm{e}^{2x}}{2w}\right)\theta_{\mathrm{e}^x/w}(t),$$

which implies that $(B^{(2\delta/\sigma^2)}_{\sigma^2 t/4},\ A^{(2\delta/\sigma^2)}_{\sigma^2 t/4})$ has the joint density function

$$\exp\left\{\frac{2\delta}{\sigma^2}x - \frac{\delta^2 t}{2\sigma^2}\right\} \frac{1}{w}\exp\left(-\frac{1+\mathrm{e}^{2x}}{2w}\right)\theta_{\mathrm{e}^x/w}\left(\frac{\sigma^2 t}{4}\right) = \tilde{g}_t(x,w),$$
$$-\infty < x < \infty, \quad w > 0.$$

Thus, by (3.2) and (3.4), we know that $(\exp\{R_t\}, \int_0^t \exp\{R_s\}\,\mathrm{d}s)$ has the following joint density function:

$$g_t(x,w) = \frac{\sigma^2}{8x}\tilde{g}_t\left(\frac{\ln x}{2}, \frac{\sigma^2 w}{4}\right), \qquad x > 0, \quad w > 0,$$

which implies that (3.3) holds. □

Theorem 3.1. *$\Phi_\alpha(u)$ satisfies the following integral equation*:

$$\begin{aligned}\Phi_\alpha(u) &= \int_0^\infty \lambda \mathrm{e}^{-(\lambda+\alpha)t}\int_0^\infty\int_0^\infty g_t(x,w)\int_0^{ux+cw}\Phi_\alpha(ux+cw-y)\,\mathrm{d}F(y)\,\mathrm{d}x\,\mathrm{d}w\,\mathrm{d}t \\ &\quad + \int_0^\infty \lambda \mathrm{e}^{-(\lambda+\alpha)t}\int_0^\infty\int_0^\infty g_t(x,w)A(ux+cw)\,\mathrm{d}x\,\mathrm{d}w\,\mathrm{d}t,\end{aligned} \tag{3.5}$$

where

$$A(u) = \int_u^\infty g(u, y-u)\,\mathrm{d}F(y). \tag{3.6}$$

Proof. Conditioning on $T_1 = t$, $Y_1 = y$, $\exp\{R_t\} = x$, and $\int_0^t \exp\{R_s\}\,\mathrm{d}s = w$, if $y \leqslant ux + cw$, then, ruin does not occur, $X_{T_1} = ux + cw - y$, and $\Phi_\alpha(ux + cw - y)$ is the expected discounted value at time t, hence $e^{-\alpha t}\Phi_\alpha(ux+cw-y)$ gives the expected discounted value at time 0. If $y > ux + cw$, then ruin occurs with $T = T_1 = t$, $X_{T-} = ux + cw$, and $|X_T| = y - (ux + cw)$. Thus, noting that T_1, Y_1, and $(\exp\{R_t\}, \int_0^t \exp\{R_s\}\,ds)$ are independent, we have

$$
\begin{aligned}
&\Phi_\alpha(u)\\
&\quad = \int_0^\infty \lambda e^{-\lambda t} E\left\{g(X_{T-}, |X_T|)e^{-\alpha T} I(T<\infty) | Y_1 = y,\ e^{R_t} = x,\right.\\
&\qquad \left.\int_0^t e^{R_s}\,ds = w,\ T_1 = t\right\} dF(y) g_t(x,w)\,dx\,dw\,dt\\
&\quad = \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty \int_0^{ux+cw} \Phi_\alpha(ux+cw-y)\,dF(y) g_t(x,w)\,dx\,dw\,dt\\
&\qquad + \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty \int_{ux+cw}^\infty g\{ux+cw,\ y-(ux+cw)\}\,dF(y)\\
&\qquad \times g_t(x,w)\,dx\,dw\,dt,
\end{aligned}
$$

which implies that (3.5) holds. □

Theorem 3.2. *Let $R_t = \delta t + \sigma B_t$ and $F(y)$ have a density function $f(y)$. Suppose that*

(a) *$g(x,y)$ in (1.5) is bounded in $x \geqslant 0$, $y \geqslant 0$;*
(b) *$f(y)$ is twice continuously differentiable on $[0,\infty)$ with $\int_0^\infty |f'(y)|\,dy < \infty$ and $\int_0^\infty |f''(y)|\,dy < \infty$;*
(c) *$A(y)$ is twice continuously differentiable on $[0,\infty)$ and both $A'(y)$ and $A''(y)$ are bounded on $[0,\infty)$;*
(d) *$\lambda + \alpha > 2(\delta + \sigma^2)$.*

Then $\Phi_\alpha(u)$ is twice continuously differentiable in $u \geqslant 0$ and both $\Phi'_\alpha(u)$ and $\Phi''_\alpha(u)$ are bounded in $u \geqslant 0$.

Further, if $f'''(y)$ and $A'''(y)$ exist and are continuous on $[0,\infty)$ with $\int_0^\infty |f'''(y)|\,dy < \infty$, a bounded $A'''(u)$ on $[0,\infty)$, and $\lambda+\alpha > 3\delta + 9\sigma^2/2$, then $\Phi'''_\alpha(u)$ is continuous and bounded in $u \geqslant 0$.

Proof. Let

$$
h(u,x,w) = \int_0^{ux+cw} f(ux+cw-y)\Phi_\alpha(y)\,dy. \tag{3.7}
$$

Thus,

$$
\int_0^{ux+cw} \Phi_\alpha(ux+cw-y)\,dF(y) = \int_0^{ux+cw} \Phi_\alpha(ux+cw-y) f(y)\,dy = h(u,x,w).
$$

Hence, (3.5) is re-expressed as

$$\Phi_\alpha(u) = \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty g_t(x,w) h(u,x,w)\,dx\,dw\,dt$$
$$+ \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty g_t(x,w) A(ux+cw)\,dx\,dw\,dt. \tag{3.8}$$

We first show that $\Phi_\alpha(u)$ is continuous in $u \geqslant 0$. By assumption (a), we know that $\Phi_\alpha(u)$ is bounded in $u \geqslant 0$ since $0 \leqslant \Phi_\alpha(u) \leqslant ME(I(T<\infty)) = M\Pr\{T<\infty\} \leqslant M$ if $0 \leqslant g(x,y) \leqslant M$. Thus, it is easy to see that $h(u,x,w)$ is continuous in $u \geqslant 0$ since $f(u)$ is continuous and $\Phi_\alpha(u)$ is bounded in $u \geqslant 0$. Further, h is bounded by (3.7), assumption (b), and the boundedness of $\Phi_\alpha(u)$. Thus, by (3.8) and the dominated convergence theorem, $\Phi_\alpha(u)$ is continuous in $u \geqslant 0$.

We next prove that $\Phi_\alpha(u)$ is continuously differentiable in $u \geqslant 0$. Since $\Phi_\alpha(u)$ is continuous and $f(u)$ is continuously differentiable in $u \geqslant 0$, we conclude from (3.7) that $h(u,x,w)$ is continuously differentiable in $u \geqslant 0$ with

$$h'_u(u,x,w) = f(0+)\Phi_\alpha(ux+cw) + x\int_0^{ux+cw} f'(ux+cw-y)\Phi_\alpha(y)\,dy, \tag{3.9}$$

which implies that

$$|h'_u(u,x,w)| \leqslant C_1 + xM_1 \quad \text{for some constants } C_1>0,\ M_1>0, \tag{3.10}$$

since assumption (b) and the boundedness of $\Phi_\alpha(u)$. Further, by assumption (c),

$$|xA'(ux+cw)| \leqslant xL_1 \quad \text{for some constant } L_1>0. \tag{3.11}$$

By Proposition 3.1 and assumption (d), we have

$$\int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty g_t(x,w)(C_1+xM_1)\,dx\,dw\,dt$$
$$= \lambda C_1 \int_0^\infty e^{-(\lambda+\alpha)t}\,dt + \lambda M_1 \int_0^\infty e^{-(\lambda+\alpha)t} E e^{R_t}\,dt$$
$$= \frac{\lambda C_1}{\lambda+\alpha} + \lambda M_1 \int_0^\infty e^{-(\lambda+\alpha)t} e^{\delta t+\sigma^2 t/2}\,dt = \frac{\lambda C_1}{\lambda+\alpha} + \frac{\lambda M_1}{\lambda+\alpha-\delta-\sigma^2/2} < \infty$$

and

$$\int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty g_t(x,w) x L_1\,dx\,dw\,dt = \lambda L_1 \int_0^\infty e^{-(\lambda+\alpha)t} E e^{R_t}\,dt < \infty.$$

Thus, by (3.8)

$$\Phi'_\alpha(u) = \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty h'_u(u,x,w) g_t(x,w)\,dx\,dw\,dt$$
$$+ \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty x g_t(x,w) A'(ux+cw)\,dx\,dw\,dt \tag{3.12}$$

is continuous in $u \geqslant 0$ and

$$|\Phi'_\alpha(u)| \leqslant \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty (C_1 + xM_1) g_t(x,w)\,dx\,dw\,dt$$
$$+ \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty x g_t(x,w) L_1\,dx\,dw\,dt < \infty,$$

which implies that $\Phi'_\alpha(u)$ is bounded in $u \geqslant 0$.

We now prove that $\Phi_\alpha(u)$ is twice continuously differentiable. By (3.9),

$$h''_u(u,x,w) = x f(0+)\Phi'_\alpha(ux+cw) + x f'(0+)\Phi_\alpha(ux+cw)$$
$$+ x^2 \int_0^{ux+cw} f''(ux+cw-y)\Phi_\alpha(y)\,dy,$$

which, together with assumption (b) and the boundedness of $\Phi_\alpha(u)$ and $\Phi'_\alpha(u)$, implies that

$$|h''_u(u,x,w)| \leqslant xC_2 + x^2 M_2 \quad \text{for some constants } C_2 > 0,\ M_2 > 0. \tag{3.13}$$

Further, by assumption (c),

$$|x^2 A''(ux+cw)| \leqslant x^2 L_2 \quad \text{for some constant } L_2 > 0. \tag{3.14}$$

By Proposition 3.1 and assumption (d), we have

$$\int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty g_t(x,w)(xC_2 + x^2M_2)\,dx\,dw\,dt$$
$$= \lambda C_2 \int_0^\infty e^{-(\lambda+\alpha)t} E e^{R_t}\,dt + \lambda M_2 \int_0^\infty e^{-(\lambda+\alpha)t} E e^{2R_t}\,dt$$
$$= \lambda C_2 \int_0^\infty e^{-(\lambda+\alpha)t} e^{\delta t + \sigma^2 t/2}\,dt + \lambda M_2 \int_0^\infty e^{-(\lambda+\alpha)t} e^{2\delta t + 2\sigma^2 t}\,dt$$
$$= \frac{\lambda C_2}{\lambda + \alpha - \delta - \sigma^2/2} + \frac{\lambda C_2}{\lambda + \alpha - 2\delta - 2\sigma^2} < \infty$$

and

$$\int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty g_t(x,w) x^2 L_2\,dx\,dw\,dt = \lambda L_2 \int_0^\infty e^{-(\lambda+\alpha)t} E e^{2R_t}\,dt < \infty.$$

Thus, by (3.12)

$$\Phi''_\alpha(u) = \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty h''_u(u,x,w) g_t(x,w)\,dx\,dw\,dt$$
$$+ \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty x^2 g_t(x,w) A''(ux+cw)\,dx\,dw\,dt$$

is continuous in $u \geqslant 0$ and by (3.13) and (3.14)

$$|\Phi_\alpha''(u)| \leqslant \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty (xC_2 + x^2 M_2) g_t(x,w)\,dx\,dw\,dt$$
$$+ \int_0^\infty \lambda e^{-(\lambda+\alpha)t} \int_0^\infty \int_0^\infty x^2 g_t(x,w) L_2\,dx\,dw\,dt < \infty,$$

which implies that $\Phi_\alpha''(u)$ is bounded in $u \geqslant 0$.

Further, similarly, we can show that $\Phi_\alpha'''(u)$ is continuous and bounded in $u \geqslant 0$ if the additional conditions hold, where the condition $\lambda + \alpha > 3\delta + 9\sigma^2/2$ guarantees

$$\int_0^\infty e^{-(\lambda+\alpha)t} E e^{3R_t}\,dt = \int_0^\infty e^{-(\lambda+\alpha)t} e^{3\delta t + 9\sigma^2 t/2}\,dt < \infty. \qquad \Box$$

Remark 3.1. Theorem 2.3 of Wang and Wu (2001) gives similar conditions for twice differentiability of the non-ruin probability $1-\psi(u)$ to those in Theorem 3.2. However, there are some mistakes in condition (iii) of Theorem 2.3 and (2.10) in Wang and Wu (2001). In fact, using notation and (2.3) of Wang and Wu (2001), it is obvious that $E[\exp\{-2B_{S_\theta}\}]$ in (2.10) of Wang and Wu (2001) should be $E[\exp\{-2B_{S_\theta}^{(v)}\}]$ with

$$E e^{-2B_{S_\theta}^{(v)}} = \int_0^\infty \frac{\theta^2}{2} e^{-\theta^2 t/2} E e^{-2B_t^{(v)}}\,dt = \int_0^\infty \frac{\theta^2}{2} e^{-\theta^2 t/2} E e^{-2vt - 2B_t}\,dt$$
$$= \int_0^\infty \frac{\theta^2}{2} e^{-\theta^2 t/2} e^{-2vt+2t}\,dt = \int_0^\infty \frac{\theta^2}{2} e^{-(4t/\sigma^2)(\lambda_0 - r)}\,dt,$$

which should replace the integral in (2.10) of Wang and Wu (2001).

Similarly, when proving twice differentiability of the non-ruin probability in Wang and Wu (2001), one needs the condition of $E e^{-4B_{S_\theta}^{(v)}} < \infty$. However,

$$E e^{-4B_{S_\theta}^{(v)}} = \int_0^\infty \frac{\theta^2}{2} e^{-\theta^2 t/2} E e^{-4B_t^{(v)}}\,dt = \int_0^\infty \frac{\theta^2}{2} e^{-\theta^2 t/2} E e^{-4vt - 4B_t}\,dt$$
$$= \int_0^\infty \frac{\theta^2}{2} e^{-\theta^2 t/2} e^{-4vt+8t}\,dt = \int_0^\infty \frac{\theta^2}{2} e^{-(4t/\sigma^2)(\lambda_0 - 2r - \sigma^2)}\,dt.$$

Hence, condition (iii) of Theorem 2.3 in Wang and Wu (2001) should be replaced by $\lambda_0 > 2r + \sigma^2$ that will guarantee both $E\exp\{-2B_{S_\theta}^{(v)}\} < \infty$ and $E\exp\{-4B_{S_\theta}^{(v)}\} < \infty$ hold.

With $\delta = r - \sigma^2/2$, the model in Wang and Wu (2001) is the same as that in this paper. Thus, Theorem 2.3 of Wang and Wu (2001) can be obtained from Theorem 3.2 by setting $g(x,y) = 1$ and $\alpha = 0$ with $A(y) = \bar{F}(y)$.

Further, the conditions on $f(y)$ in Theorem 3.2 can be satisfied by many claim size distributions such as exponential distributions, mixed exponential distributions, Erlang distributions, Pareto distributions with finite means, and lognormal distributions. The conditions on $A(y)$ in Theorem 3.2 can be satisfied by many interesting examples, which are given in Section 5.

4. Integro-differential equations

In this section, using a differential argument and moments of exponential functionals of Brownian motions given in the Appendix we derive an integro-differential equation for $\Phi_\alpha(u)$. The differential argument is a common method used in ruin theory. See Grandell (1991) for the method used in the compound Poisson surplus process and Dufresne and Gerber (1991) for it used in the compound Poisson surplus process perturbed by a diffusion.

Theorem 4.1. *Under the conditions of Theorem* 3.2, $\Phi_\alpha(u)$ *satisfies the following integro-differential equation*:

$$\frac{\sigma^2 u^2}{2}\Phi''_\alpha(u) + \left(c + \left(\delta + \frac{\sigma^2}{2}\right)u\right)\Phi'_\alpha(u) + \lambda A(u) = (\lambda+\alpha)\Phi_\alpha(u) - \lambda\int_0^u \Phi_\alpha(u-y)\,\mathrm{d}F(y) \tag{4.1}$$

and the following boundary conditions:

$$\begin{aligned}&\Phi_\alpha(+\infty) = 0,\\ &c\Phi'_\alpha(0^+) + \lambda A(0) = (\lambda+\alpha)\Phi_\alpha(0).\end{aligned} \tag{4.2}$$

Proof. Let

$$h(t) = u(\exp\{R_t\} - 1) + c\int_0^t \exp\{R_s\}\,\mathrm{d}s.$$

Consider the surplus process X_t in a very short time interval $(0,\Delta t]$. Since $N(t)$ is a Poisson process, there are four possible cases in $(0,\Delta t]$ as follows:

- (i) no claim in $(0,\Delta t]$, thus $X_{\Delta t} = u\exp\{R_{\Delta t}\} + c\int_0^{\Delta t}\exp\{R_s\}\,\mathrm{d}s = u + h(\Delta t)$;
- (ii) one claim in $(0,\Delta t]$, and the amount of the claim $y \leqslant u + h(\Delta t)$, i.e. the claim does not cause ruin, thus $X_{\Delta t} = u\exp\{R_{\Delta t}\} + c\int_0^{\Delta t}\exp\{R_s\}\,\mathrm{d}s - y = u + h(\Delta t) - y$;
- (iii) one claim in $(0,\Delta t]$, and the amount of the claim $y > u + h(\Delta t)$, i.e. the claim causes ruin, thus, $X_{\Delta t-} = u + h(\Delta t)$ and $|X_{\Delta t}| = y - (u + h(\Delta t))$;
- (iv) more than one claim in $(0,\Delta t]$.

Thus, considering the four cases, we have

$$\begin{aligned}\Phi_\alpha(u) = {}& (1-\lambda\Delta t)\mathrm{e}^{-\alpha\Delta t}E[\Phi_\alpha(u+h(\Delta t))] + \lambda\Delta t\mathrm{e}^{-\alpha\Delta t}\\ &\times E\left[\int_0^{u+h(\Delta t)}\Phi_\alpha(u+h(\Delta t)-y)\,\mathrm{d}F(y)\right]\\ &+ \lambda\Delta t\mathrm{e}^{-\alpha\Delta t}E\left[\int_{u+h(\Delta t)}^\infty g(u+h(\Delta t), y-[u+h(\Delta t)])\,\mathrm{d}F(y)\right] + o(\Delta t)\end{aligned}$$

$$= (1-\lambda\Delta t)\mathrm{e}^{-\alpha\Delta t}E[\Phi_\alpha(u+h(\Delta t))] + \lambda\Delta t\mathrm{e}^{-\alpha\Delta t}$$
$$\times E\left[\int_0^{u+h(\Delta t)} \Phi_\alpha(u+h(\Delta t)-y)\,\mathrm{d}F(y)\right]$$
$$+ \lambda\Delta t\mathrm{e}^{-\alpha\Delta t}E[A(u+h(\Delta t))] + o(\Delta t). \tag{4.3}$$

By Taylor's expansion, we have

$$E[\Phi_\alpha(u+h(\Delta t))]$$
$$= \Phi_\alpha(u) + \Phi'_\alpha(u)E[h(\Delta t)] + \Phi''_\alpha(u)E\left[\frac{h^2(\Delta t)}{2!}\right] + E\left[\Phi'''_\alpha(\tilde{u})\frac{h^3(\Delta t)}{3!}\right], \tag{4.4}$$

where $\tilde{u}$ is between u and $u+h(\Delta t)$.

However, by (A.9) and $|\Phi'''_\alpha(\tilde{u})| < M$ for some constant $M>0$, we have

$$0 \leqslant \frac{1}{\Delta t}\left|E\left[\Phi'''_\alpha(\tilde{u})\frac{h^3(\Delta t)}{3!}\right]\right| \leqslant \left(\frac{M}{6}\right)\frac{E[h^3(\Delta t)]}{\Delta t} \to 0 \quad \text{as } \Delta t \to 0. \tag{4.5}$$

Thus, by dividing Δt on both sides of (4.3), letting $\Delta t \to 0$, and using (4.4), (4.5), (A.8), and

$$\lim_{\Delta t\to 0} \frac{1-(1-\lambda\Delta t)\exp\{-\alpha\Delta t\}}{\Delta t} = \lambda+\alpha,$$

we obtain

$$(\lambda+\alpha)\Phi_\alpha(u) = \left(c + \left(\delta + \frac{\sigma^2}{2}\right)u\right)\Phi'_\alpha(u) + \frac{\sigma^2u^2}{2}\Phi''_\alpha(u)$$
$$+\lambda\int_0^u \Phi_\alpha(u-y)\,\mathrm{d}F(y) + \lambda A(u),$$

which yields (4.1).

The boundary conditions in (4.2) follow from $0 \leqslant \Phi_\alpha(u) \leqslant M\psi(u)$ if $0 \leqslant g(x,y) \leqslant M$, $\psi(+\infty)=0$, and letting $u \downarrow 0$ in (4.1). □

Corollary 4.1. *Under the conditions of Theorem* 4.1, *if F is an exponential distribution with a density function $f(x)=\beta\mathrm{e}^{-\beta x}$, $x>0$, $\beta>0$, then $\Phi_\alpha(u)$ satisfies the following third-order differential equation*:

$$\frac{\sigma^2u^2}{2}\Phi'''_\alpha(u) + a(u)\Phi''_\alpha(u) + b(u)\Phi'_\alpha(u) - \alpha\beta\Phi_\alpha(u) + \lambda K(u) = 0 \tag{4.6}$$

and the following boundary conditions:

$$\Phi_\alpha(+\infty) = 0,$$
$$c\Phi'_\alpha(0^+) + \lambda A(0) = (\lambda+\alpha)\Phi_\alpha(0),$$
$$c\Phi''_\alpha(0^+) + \left(-\lambda - \alpha + \delta + \frac{\sigma^2}{2} + \beta c\right)\Phi'_\alpha(0^+) = \alpha\beta\Phi_\alpha(0) - \lambda K(0), \tag{4.7}$$

where

$$K(u) = A'(u) + \beta A(u),$$

$$a(u) = c + \left(\delta + \frac{3\sigma^2}{2}\right) u + \frac{\beta^2\sigma^2}{2} u^2$$

and

$$b(u) = -\lambda - \alpha + \delta + \frac{\sigma^2}{2} + \beta c + \beta \left(\delta + \frac{\sigma^2}{2}\right) u.$$

Proof. When F is exponential, (4.1) is re-expressed as

$$(\lambda+\alpha)\Phi_\alpha(u) - \left(c + \left(\delta + \frac{\sigma^2}{2}\right) u\right) \Phi'_\alpha(u) - \frac{\sigma^2 u^2}{2} \Phi''_\alpha(u) - \lambda A(u)$$
$$= \lambda\beta e^{-\beta u} \int_0^u e^{\beta y} \Phi_\alpha(y)\, dy. \tag{4.8}$$

By taking the derivative with respect to u on both sides of (4.8), we obtain

$$(\lambda+\alpha)\Phi'_\alpha(u) - \left(\delta + \frac{\sigma^2}{2}\right) \Phi'_\alpha(u) - \left(c + \left(\delta + \frac{\sigma^2}{2}\right) u\right) \Phi''_\alpha(u) - \sigma^2 u \Phi''_\alpha(u)$$
$$-\frac{\sigma^2 u^2}{2} \Phi'''_\alpha(u) - \lambda A'(u)$$
$$= \left(\lambda + \alpha - \delta - \frac{\sigma^2}{2}\right) \Phi'_\alpha(u) - \left(c + \left(\delta + \frac{3\sigma^2}{2}\right) u\right) \Phi''_\alpha(u)$$
$$-\frac{\sigma^2 u^2}{2} \Phi'''_\alpha(u) - \lambda A'(u)$$
$$= -\beta \left[\lambda\beta e^{-\beta u} \int_0^u e^{\beta y} \Phi_\alpha(y)\, dy\right] + \lambda\beta\Phi_\alpha(u),$$

which, together with (4.8), gives

$$\left(\lambda + \alpha - \delta - \frac{\sigma^2}{2}\right) \Phi'_\alpha(u) - \left(c + \left(\delta + \frac{3\sigma^2}{2}\right) u\right) \Phi''_\alpha(u)$$
$$-\frac{\sigma^2 u^2}{2} \Phi'''_\alpha(u) - \lambda A'(u)$$
$$= -\beta \left[(\lambda+\alpha)\Phi_\alpha(u) - \left(c + \left(\delta + \frac{\sigma^2}{2}\right) u\right) \Phi'_\alpha(u)\right.$$
$$\left. -\frac{\sigma^2 u^2}{2} \Phi''_\alpha(u) - \lambda A(u)\right] + \lambda\beta\Phi_\alpha(u),$$

which implies that (4.6) holds.

The boundary conditions in (4.7) follow from (4.2) and letting $u \downarrow 0$ in (4.6). □

The third-order differential equation, together with the boundary conditions, in Corollary 4.1 enables us to obtain explicit solutions or asymptotic formulae for ruin quantities when the claim sizes are exponentially distributed. For instance, see Paulsen and Gjessing (1997) for the ruin probability with $\sigma = 0$; Frolova et al. (2002) for the ruin probability with $\sigma > 0$; and Cai and Dickson (2002) and references therein for other ruin quantities with $\sigma = 0$.

5. Applications

In this section, we give some examples to illustrate applications of the integro-differential equation derived in Section 4.

Example 5.1. Let $g(x, y)=1$ and $\alpha > 0$, we have $\Phi_\alpha(u)=E(\mathrm{e}^{-\alpha T} I(T < \infty))=E(\mathrm{e}^{-\alpha T})=q_\alpha(u)$ is the Laplace transform of the time of ruin with an initial surplus u. In this case, $A(u) = \bar{F}(u)$. Thus by (4.1), $q_\alpha(u)$ satisfies

$$\frac{\sigma^2 u^2}{2} q_\alpha''(u) + \left(c + \left(\delta + \frac{\sigma^2}{2}\right) u\right) q_\alpha'(u) + \lambda \bar{F}(u)$$

$$= (\lambda + \alpha) q_\alpha(u) - \lambda \int_0^u q_\alpha(u - y)\, \mathrm{d}F(y)$$

and the following boundary conditions:

$$q_\alpha(+\infty) = 0,$$

$$c q_\alpha'(0^+) + \lambda = (\lambda + \alpha) q_\alpha(0),$$

which give Theorem 2.1(ii) of Paulsen and Gjessing (1997) with $\sigma_P = \lambda_R = 0$ and $\delta = r - \frac{1}{2}\sigma_R^2$.

When F is an exponential distribution with $F'(x) = \beta \mathrm{e}^{-\beta x}$, $x > 0$, $\beta > 0$, we have $A(y) = \mathrm{e}^{-\beta y}$, hence $K(u) = A'(u) + \beta A(u) = 0$. Thus, by (4.6), $q_\alpha(u)$ satisfies

$$\frac{\sigma^2 u^2}{2} q_\alpha'''(u) + \left\{c + \left(\delta + \frac{3\sigma^2}{2}\right) u + \frac{\beta^2 \sigma^2}{2} u^2\right\} q_\alpha''(u)$$

$$+ \left\{-\lambda - \alpha + \delta + \frac{\sigma^2}{2} + \beta c + \beta\left(\delta + \frac{\sigma^2}{2}\right) u\right\} q_\alpha'(u) - \alpha\beta q_\alpha(u) = 0$$

and the following boundary conditions:

$$q_\alpha(+\infty) = 0,$$

$$c q_\alpha'(0^+) + \lambda = (\lambda + \alpha) q_\alpha(0),$$

$$c q_\alpha''(0^+) + \left(-\lambda - \alpha + \delta + \frac{\sigma^2}{2} + \beta c\right) q_\alpha'(0^+) = \alpha\beta q_\alpha(0),$$

which give (2.11) of Paulsen and Gjessing with $\sigma_P = \sigma_R = 0$ and $\delta = r - \frac{1}{2}\sigma_R^2$, and recover (2.14) of Paulsen and Gjessing (1997) with $\sigma = 0$. □

Example 5.2. *Let* $g(x, y) = 1$ and $\alpha = 0$, we have $\Phi_\alpha(u) = E(I(T < \infty)) = \psi(u)$ and $A(u) = \bar{F}(u)$. Thus, by (4.1), $\psi(u)$ satisfies

$$\frac{\sigma^2 u^2}{2}\psi''(u) + \left(c + \left(\delta + \frac{\sigma^2}{2}\right)u\right)\psi'(u) + \lambda\bar{F}(u)$$
$$= \lambda\psi(u) - \lambda\int_0^u \psi(u-y)\,\mathrm{d}F(y)$$

and the following boundary conditions:

$$\psi(+\infty) = 0,$$
$$c\psi'(0^+) = \lambda[\psi(0) - 1],$$

which recover Theorem 2.1(i) of Paulsen and Gjessing (1997) with $\sigma_P = \lambda_R = 0$ and $\delta = r - \frac{1}{2}\sigma_R^2$, and imply Theorem 2.4 of Wang and Wu (2001) with the non-ruin probability $\phi(u) = 1 - \psi(u)$ and $\delta = r - \frac{1}{2}\sigma^2$.

When F is an exponential distribution with $F'(x) = \beta\mathrm{e}^{-\beta x}$, $x > 0$, $\beta > 0$, we have $K(u) = 0$. Thus, by (4.6), we know that $\psi(u)$ satisfies

$$\frac{\sigma^2 u^2}{2}\psi'''(u) + \left\{c + \left(\delta + \frac{3\sigma^2}{2}\right)u + \frac{\beta^2\sigma^2}{2}u^2\right\}\psi''(u)$$
$$+ \left\{-\lambda + \delta + \frac{\sigma^2}{2} + \beta c + \beta\left(\delta + \frac{\sigma^2}{2}\right)u\right\}\psi'(u) = 0$$

and the following boundary conditions:

$$\psi(+\infty) = 0,$$
$$c\psi'(0^+) = \lambda[\psi(0) - 1],$$
$$c\psi''(0^+) + \left(-\lambda + \delta + \frac{\sigma^2}{2} + \beta c\right)\psi'(0^+) = 0,$$

which imply (2.20) and (2.21) of Wang and Wu (2001) with the non-ruin probability $\phi(u) = 1 - \psi(u)$ and $\delta = r - \frac{1}{2}\sigma^2$.

Example 5.3. Let $g(x_1, x_2) = I(x_2 \leqslant y)$ and $\alpha = 0$, then

$$\Phi_\alpha(u) = \Pr\{|X_T| \leqslant y,\ T < \infty\} = G(u, y)$$

is the distribution function of the deficit at ruin. In this case, $A(u) = \bar{F}(u) - \bar{F}(u + y)$. Thus, (4.1) implies that $G(u, y)$ satisfies the following integro-differential equation:

$$\frac{\sigma^2 u^2}{2}\, G''_u(u,y) + \left(c + \left(\delta + \frac{\sigma^2}{2}\right) u\right) G'_u(u,y) + \lambda\{\bar{F}(u) - \bar{F}(u+y)\}$$
$$= \lambda G(u,y) - \lambda \int_0^u G(u-t,y)\,\mathrm{d}F(t)$$

and the following boundary conditions:

$$G(+\infty, y) = 0,$$
$$cG'_u(0^+, y) + \lambda F(y) = \lambda G(0,y),$$

which give Theorem 3.4 of Wang and Wu (2001) with $\delta = r - \frac{1}{2}\sigma^2$.

When F is an exponential distribution with $F'(x) = \beta e^{-\beta x}$, $x > 0$, $\beta > 0$, we have $A(u) = e^{-\beta u}(1 - e^{-\beta y})$, hence $K(u) = 0$. Thus, Corollary 4.1 gives (3.9) and (3.10) of Wang and Wu (2001) with $\delta = r - \frac{1}{2}\sigma^2$.

Example 5.4. Let $g(x_1, x_2) = \exp\{-rx_2\}$, $r \geqslant 0$, and $\alpha = 0$, then

$$\Phi_\alpha(u) = E[\exp\{-r|X_T|\} I(T < \infty)] = \tilde{G}(u,r)$$

is the Laplace transform of the deficit at ruin. In this case,

$$A(u) = \int_u^\infty \exp\{-r(x-u)\} f(x)\,\mathrm{d}x.$$

Thus, by (4.1), $\tilde{G}(u,r)$ satisfies the following integro-differential equation:

$$\frac{\sigma^2 u^2}{2}\, \tilde{G}''_u(u,r) + \left(c + \left(\delta + \frac{\sigma^2}{2}\right) u\right) \tilde{G}'_u(u,r) + \lambda \int_u^\infty \exp\{-r(x-u)\} f(x)\,\mathrm{d}x$$
$$= \lambda \tilde{G}(u,r) - \lambda \int_0^u \tilde{G}(u-y,r)\,\mathrm{d}F(y)$$

and the following boundary conditions:

$$\tilde{G}(+\infty, r) = 0,$$
$$c\tilde{G}'_u(0^+, r) + \lambda \tilde{F}(r) = \lambda \tilde{G}(0,r),$$

where $\tilde{F}(r) = \int_0^\infty e^{-rx}\,\mathrm{d}F(x)$ is the Laplace transform of F.

When F is an exponential distribution with $F'(x) = \beta e^{-\beta x}$, $x > 0$, $\beta > 0$, we have $A(u) = (\beta/(r+\beta)) \exp\{-\beta u\}$ and $K(u) = 0$. Thus, by (4.6), we know that $\tilde{G}(u,r)$ satisfies

$$\frac{\sigma^2 u^2}{2}\, \tilde{G}'''_u(u,r) + \left\{c + \left(\delta + \frac{3\sigma^2}{2}\right) u + \frac{\beta^2 \sigma^2}{2} u^2\right\} \tilde{G}''_u(u,r)$$
$$+ \left\{-\lambda + \delta + \frac{\sigma^2}{2} + \beta c + \beta\left(\delta + \frac{\sigma^2}{2}\right) u\right\} \tilde{G}'_u(u,r) = 0$$

and the following boundary conditions:

$$\tilde{G}(+\infty, r) = 0,$$

$$c\tilde{G}_u'(0^+, r) + \lambda\tilde{F}(r) = \lambda\tilde{G}(0, r),$$

$$c\tilde{G}_u''(0^+, r) + \left(-\lambda + \delta + \frac{\sigma^2}{2} + \beta c\right)\tilde{G}_u'(0^+, r) = 0.$$

Example 5.5. Let $g(x_1, x_2) = \exp\{-r(x_1 + x_2)\}$, $r \geqslant 0$, and $\alpha = 0$, then

$$\Phi_\alpha(u) = E[\exp\{-r(X_{T-} + |X_T|)\}I(T < \infty)] = \tilde{D}(u, r)$$

is the Laplace transform of the amount of claim-causing ruin. In this case, $A(u) = \int_u^\infty e^{-rx} f(x)\,dx$. Thus, by (4.1), $\tilde{D}(u, r)$ satisfies the following integro-differential equation:

$$\frac{\sigma^2 u^2}{2}\tilde{D}_u''(u, r) + \left(c + \left(\delta + \frac{\sigma^2}{2}\right)u\right)\tilde{D}_u'(u, r) + \lambda\int_u^\infty \exp\{-rx\}f(x)\,dx$$

$$= \lambda\tilde{D}(u, r) - \lambda\int_0^u \tilde{D}(u - y, r)\,dF(y)$$

and the following boundary conditions:

$$\tilde{D}(+\infty, r) = 0,$$

$$c\tilde{D}_u'(0^+, r) + \lambda\tilde{F}(r) = \lambda\tilde{D}(0, r).$$

When F is an exponential distribution with $F'(x) = \beta e^{-\beta x}$, $x > 0$, $\beta > 0$, we have $A(u) = (\beta/(r+\beta))\exp\{-(r+\beta)u\}$, hence $K(u) = A'(u) + \beta A(u) = (-r\beta/(r+\beta))\exp\{-(r+\beta)u\}$. Thus, by (4.6), we know that $\tilde{D}(u, r)$ satisfies

$$\frac{\sigma^2 u^2}{2}\tilde{D}_u'''(u, r) + \left\{c + \left(\delta + \frac{3\sigma^2}{2}\right)u + \frac{\beta^2\sigma^2}{2}u^2\right\}\tilde{D}_u''(u, r)$$

$$+ \left\{-\lambda + \delta + \frac{\sigma^2}{2} + \beta c + \beta\left(\delta + \frac{\sigma^2}{2}\right)u\right\}\tilde{D}_u'(u, r) = \frac{\lambda r\beta}{r + \beta}\exp\{-(r + \beta)u\}$$

and the following boundary conditions:

$$\tilde{D}(+\infty, r) = 0,$$

$$c\tilde{D}_u'(0^+, r) + \lambda\tilde{F}(r) = \lambda\tilde{D}(0, r),$$

$$c\tilde{D}_u''(0^+, r) + \left(-\lambda + \delta + \frac{\sigma^2}{2} + \beta c\right)\tilde{D}_u'(0^+, r) = \frac{\lambda r\beta}{r + \beta}.$$

Acknowledgements

The author wishes to thank the referee for useful comments. This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

Appendix

Let $R_t = \delta + \sigma B_t$. We give some results about moments of $\exp\{R_t\}$ and $\int_0^t \exp\{R_s\}\,\mathrm{d}s$ using the results of Yor (1992). It is obvious that

$$E(\exp\{R_t\} - 1) = E(\exp\{\delta t + \sigma B_t\}) - 1 = \exp\left\{\left(\delta + \frac{\sigma^2}{2}\right) t\right\} - 1$$

and

$$E(\exp\{R_t\} - 1)^2 = \exp\{2\delta t + 2\sigma^2 t\} - 2\exp\left\{\delta t + \frac{\sigma^2}{2} t\right\} + 1,$$

which give

$$\lim_{t\to 0} \frac{E(\exp\{R_t\}) - 1}{t} = \delta + \frac{\sigma^2}{2} \quad \text{and} \quad \lim_{t\to 0} \frac{E(\exp\{R_t\} - 1)^2}{t} = \sigma^2. \tag{A.1}$$

By (4.d′) of Yor (1992), we obtain

$$E\left[\int_0^t \exp\{R_s\}\,\mathrm{d}s\right] = \frac{\exp\{(\delta + \sigma^2/2)t\} - 1}{\delta + \sigma^2/2} \tag{A.2}$$

and

$$E\left(\int_0^t \exp\{R_s\}\,\mathrm{d}s\right)^2$$

$$= \frac{2[(2\delta + 3\sigma^2) - 4(\delta + \sigma^2)\exp\{(\delta + \sigma^2/2)t\} + (2\delta + \sigma^2)\exp\{2(\delta + \sigma^2)t\}]}{(\delta + \sigma^2)(2\delta + \sigma^2)(2\delta + 3\sigma^2)},$$

which imply that

$$\lim_{t\to 0} \frac{E(\int_0^t \exp\{R_s\}\,\mathrm{d}s)}{t} = 1 \quad \text{and} \quad \lim_{t\to 0} \frac{E(\int_0^t \exp\{R_s\}\,\mathrm{d}s)^2}{t} = 0. \tag{A.3}$$

By $(4.\mathrm{c})_v$ and Lemma 2 of Yor (1992), we have

$$E\left[\exp\{B_t^{(v)}\} \int_0^t \exp\{B_s^{(v)}\}\,\mathrm{d}s\right]$$

$$= E\left[\exp\{B_t^{(v)}\}\left(-\frac{2}{2(1+v)+1} + \frac{2\exp\{B_t^{(v)}\}}{2(1+v)+1}\right)\right]$$

$$= \frac{2}{2v+3}\left[\exp\{2vt + 2t\} - \exp\left\{vt + \frac{t}{2}\right\}\right]$$

$$= \exp\left\{\left(v + \frac{1}{2}\right)t\right\} \frac{\exp\{(v + 3/2)t\} - 1}{v + 3/2}. \tag{A.4}$$

By the scaling property of Brownian motions, we know that $\tilde{B}_t = \sigma B_{t/\sigma^2}$ is also a Brownian motion. Thus, by letting $s = x/\sigma^2$ and noting $\tilde{B}_{\sigma^2 t} = \sigma B_t$, we obtain that

$$\begin{aligned}
&E\left[\exp\{R_t\}\int_0^t \exp\{R_s\}\,\mathrm{d}s\right]\\
&\quad = E\left[\exp\{\delta t + \sigma B_t\}\int_0^t \exp\{\delta s + \sigma B_s\}\,\mathrm{d}s\right]\\
&\quad = \frac{1}{\sigma^2}\,E\left[\exp\{\delta t + \sigma B_t\}\int_0^{\sigma^2 t} \exp\left\{(\delta/\sigma^2)x + \sigma B_{x/\sigma^2}\right\}\,\mathrm{d}x\right]\\
&\quad = \frac{1}{\sigma^2}\,E\left[\exp\{(\delta/\sigma^2)(\sigma^2 t) + \tilde{B}_{\sigma^2 t}\}\int_0^{\sigma^2 t} \exp\{(\delta/\sigma^2)x + \tilde{B}_x\}\,\mathrm{d}x\right]. \qquad (A.5)
\end{aligned}$$

Therefore, by (A.5) and (A.4), we get

$$\begin{aligned}
&E\left[\exp\{R_t\}\int_0^t \exp\{R_s\}\,\mathrm{d}s\right]\\
&\quad = \left(\frac{1}{\sigma^2}\right)\exp\left\{\left(\frac{\delta}{\sigma^2}+\frac{1}{2}\right)(\sigma^2 t)\right\}\frac{\exp\{(\delta/\sigma^2+3/2)(\sigma^2 t)\}-1}{\delta/\sigma^2+3/2}\\
&\quad = \exp\left\{\left(\delta+\frac{\sigma^2}{2}\right)t\right\}\frac{\exp\{(\delta+3\sigma^2/2)t\}-1}{\delta+3\sigma^2/2}. \qquad (A.6)
\end{aligned}$$

By (A.2) and (A.6), we have

$$\begin{aligned}
&E\left[(\exp\{R_t\}-1)\int_0^t \exp\{R_s\}\,\mathrm{d}s\right]\\
&\quad = \exp\left\{\left(\delta+\frac{\sigma^2}{2}\right)t\right\}\frac{\exp\{(\delta+3\sigma^2/2)t\}-1}{\delta+3\sigma^2/2} - \frac{\exp\{(\delta+\sigma/2)t\}-1}{\delta+\sigma^2/2},
\end{aligned}$$

which implies that

$$\lim_{t\to 0}\frac{E[(\exp\{R_t\}-1)\int_0^t \exp\{R_s\}\,\mathrm{d}s]}{t} = 0. \qquad (A.7)$$

Thus, recalling $h(t) = u(\exp\{R_t\}-1) + c\int_0^t \exp\{R_s\}\,\mathrm{d}s$ and by (A.1), (A.2), and (A.6), we have

$$\lim_{t\to 0}\frac{Eh(t)}{t} = u\left(\delta+\frac{\sigma^2}{2}\right)+c \quad \text{and} \quad \lim_{t\to 0}\frac{Eh^2(t)}{t} = \sigma^2 u^2. \qquad (A.8)$$

Similarly, we can prove that

$$\lim_{t\to 0}\frac{Eh^3(t)}{t} = 0. \qquad (A.9)$$

In fact,

$$h^3(t) = u^3(\exp\{R_t\} - 1)^3 + 3u^2c(\exp\{R_t\} - 1)^2 \int_0^t \exp\{R_s\}\,\mathrm{d}s$$
$$+3uc^2(\exp\{R_t\} - 1)\left(\int_0^t \exp\{R_s\}\,\mathrm{d}s\right)^2 + c^3\left(\int_0^t \exp\{R_s\}\,\mathrm{d}s\right)^3.$$

Using similar arguments to those for (A.7), we can show that

$$\lim_{t\to 0} \frac{E(\exp\{R_t\} - 1)^3}{t} = \lim_{t\to 0} \frac{Et[(\exp\{R_t\} - 1)^2 \int_0^t \exp\{R_s\}\,\mathrm{d}s]}{t}$$
$$= \lim_{t\to 0} \frac{E[(\exp\{R_t\} - 1)(\int_0^t \exp\{R_s\}\,\mathrm{d}s)^2]}{t}$$
$$= \lim_{t\to 0} \frac{E[(\int_0^t \exp\{R_s\}\,\mathrm{d}s)^3]}{t} = 0.$$

For example,

$$E(\exp\{R_t\} - 1)^3 = E\exp\{3R_t\} - 3E\exp\{2R_t\} + 3E\exp\{R_t\} - 1$$
$$= \exp\left\{3\delta t + \frac{9}{2}\sigma^2 t\right\} - 3\exp\{2\delta t + 2\sigma^2 t\}$$
$$+3\exp\left\{\delta t + \frac{1}{2}\sigma^2 t\right\} - 1$$

gives that $\lim_{t\to 0} \frac{E[(\exp\{R_t\}-1)^3]}{t} = 0$.

References

Asmussen, S., 2000. Ruin Probabilities. World Scientific, Singapore.

Cai, J., 2002. Ruin probabilities under dependent rates of interest. J. Appl. Probab. 39, 312–323.

Cai, J., Dickson, D.C.M., 2002. On the expected discounted penalty function at ruin of a surplus process with interest. Insurance Math. Econom. 30, 389–404.

Cai, J., Dickson, D.C.M., 2003. Upper bounds for ultimate ruin probabilities in the Sparre Andersen model with interest. Insurance Math. Econom. 32, 61–71.

Carmona, P., Petit, F., Yor, M., 1994. On exponential functionals of certain Lévy processes. Stochastics Stochastic Rep. 47, 71–101.

Dufresne, F., Gerber, H.U., 1991. Risk theory for the compound Poisson process that is perturbed by diffusion. Insurance Math. Econom. 10, 51–59.

Frolova, A., Kabanov, Y., Pergamenshchikov, S., 2002. In the insurance business risky investments are dangerous. Finance Stoch 6, 227–235.

Gerber, H.U., Shiu, E.S.W., 1997. The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. Insurance Math. Econom. 21, 129–137.

Grandell, J., 1991. Aspects of Risk Theory. Springer, New York.

Kalashnikov, V., Norberg, R., 2002. Power tailed ruin probabilities in the presence of risky investments. Stochastic Process. Appl. 98, 211–228.

Norberg, R., 1999. Ruin probabilities with assets and liabilities of diffusion type. Stochastic Process. Appl. 81, 255–269.

Paulsen, J., 1993. Risk theory in a stochastic environment. Stochastic Process. Appl. 21, 327–361.
Paulsen, J., 2002. On Cramér-like asymptotics for risk processes with stochastic return on investments. Ann. Appl. Probab. 12, 1247–1260.
Paulsen, J., Gjessing, H.K., 1997. Ruin theory with stochastic economic environment. Adv. Appl. Probab. 29, 965–985.
Rolski, T., Schmidli, V., Schmidt, V., Teugels, J.L., 1999. Stochastic Processes for Insurance and Finance. Wiley, Chichester.
Sundt, B., Teugels, J.L., 1995. Ruin estimates under interest force. Insurance Math. Econom. 16, 7–22.
Wang, G., Wu, R., 2001. Distributions for the risk process with a stochastic return on investments. Stochastic Process. Appl. 95, 329–341.
Willmot, G.E., Lin, X.S., 2001. Lundberg Approximations for Compound Distributions with Insurance Applications. Springer, New York.
Yor, M., 1992. On some exponential functionals of Brownian motion. Adv. Appl. Probab. 24, 509–531.